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The two-body problem for the metric of Nordtvedt

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Abstract. The postnewtonian n-body equation for the general metric expansion of Nordtvedt is derived. By taking $n = 2$, the advance of the periastron and the motion of the centre of mass in the two-body problem are examined. This enables the formulae for these effects to be found in the Brans–Dicke and Nordtvedt scalar–tensor theories.

1. Introduction

The idea of considering relativistic gravitational tests in terms of general metric expansions has received some attention during the past few years. This is based on the work of Schiff (1967) and others who expanded the metric of a single body in terms of the dimensionless quantity m/r where $m = GM/c^2$ is the geometrized mass of the body and r the radial distance from it. If the metric is of the form

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu, \qquad \mu, \nu = 0, 1, 2, 3,$$

with signature $(+1, -1, -1, -1)$, then the expansion is:

$$\begin{aligned} g_{00} &= 1-2\alpha\frac{m}{r}+2\beta\left(\frac{m}{r}\right)^2+\ldots, \\ g_{0k} &= 0, \qquad k = 1, 2, 3, \\ g_{kl} &= -\left(1+2\gamma\frac{m}{r}\right)\delta_{kl}+\ldots, \qquad k, l = 1, 2, 3, \end{aligned} \tag{1}$$

where α, β and γ are dimensionless constants whose values depend on the particular gravitational theory being considered. Using this expansion theoretical formulae for relativistic tests are calculated in terms of α, β, γ and other constant factors (see Nordtvedt (1968) for a summary of these). Experiments are looked upon as a means of determining the values of α, β and γ rather than as agreeing or disagreeing with a particular theory. For example, the test involving the measurement of the time delay of a radar signal reflected by a planet and passing close to the Sun determines a value for $(\alpha + \gamma)$. Recently, Shapiro and his colleagues have made an accurate determination of this expression (Shapiro *et al* 1971). For a systematic treatment of this approach to experimental relativity see Thorne and Will (1971).

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Nordtvedt (1969) has extended the above metric by writing down a general postnewtonian expansion for n moving bodies. He used this to examine an effect which does not arise in general relativity, that is, the violation of the equivalence principle by massive bodies. In the present paper the postnewtonian n-body equation of motion for Nordtvedt's metric is derived. A special case of this equation is then used to examine the two-body problem for this general metric.

2. The postnewtonian metric expansion

The generalization of equation (1) to n moving sources in the postnewtonian (ie c^{-2}) approximation given by Nordtvedt (1968, 1969) is as follows:

$$g_{00} = 1-2\alpha\sum_i \frac{m_i}{|\boldsymbol{r}-\boldsymbol{r}_i|}+2\beta\left(\sum_i \frac{m_i}{|\boldsymbol{r}-\boldsymbol{r}_i|}\right)^2+2\alpha'\sum_i\sum_{j\neq i}\frac{m_i}{|\boldsymbol{r}-\boldsymbol{r}_i|}\frac{m_j}{|\boldsymbol{r}_i-\boldsymbol{r}_j|}$$
$$+\frac{\chi}{c^2}\sum_i\frac{m_i(\boldsymbol{r}-\boldsymbol{r}_i)\cdot\boldsymbol{a}_i}{|\boldsymbol{r}-\boldsymbol{r}_i|}-\frac{4\alpha''}{c^2}\sum_i\frac{m_iv_i^2}{|\boldsymbol{r}-\boldsymbol{r}_i|}+\frac{\alpha'''}{c^2}\sum_i\frac{m_i\{(\boldsymbol{r}-\boldsymbol{r}_i)\cdot\boldsymbol{v}_i\}^2}{|\boldsymbol{r}-\boldsymbol{r}_i|^3} \tag{2a}$$

$$g_{0k} = \frac{4\Delta}{c}\sum_i\frac{m_i(\boldsymbol{v}_i)_k}{|\boldsymbol{r}-\boldsymbol{r}_i|}+\frac{4\Delta'}{c}\sum_i\frac{m_i\{(\boldsymbol{r}-\boldsymbol{r}_i)\cdot\boldsymbol{v}_i\}(\boldsymbol{r}-\boldsymbol{r}_i)_k}{|\boldsymbol{r}-\boldsymbol{r}_i|^3} \tag{2b}$$

$$g_{kl} = -\left(1+2\gamma\sum_i\frac{m_i}{|\boldsymbol{r}-\boldsymbol{r}_i|}\right)\delta_{kl}, \tag{2c}$$

where m_i, $\boldsymbol{r}_i$, $\boldsymbol{v}_i$ and $\boldsymbol{a}_i$ are respectively the geometrized mass position, velocity and acceleration of the ith source ($i = 1, \dots, n$), $v_i^2 = |\boldsymbol{v}_i|^2$, and $\chi, \alpha', \alpha'', \alpha''', \Delta$ and Δ' are further dimensionless constants taking specific values for a particular gravitational theory. Note that the α' term in equation (2a) is different from that given by Nordtvedt (1969). This is because Nordtvedt (1968) wrote the term in the two-body case as

$$2\alpha'\frac{m_1m_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_1|}+\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_2|}\right),$$

and seems to have assumed that the n-body generalization of this is

$$2\alpha'\sum_i\sum_{j\neq i}\frac{m_im_j}{|\boldsymbol{r}_i-\boldsymbol{r}_j|}\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_i|}+\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_j|}\right),$$

which is easily seen to be false. The true generalization is that given in equation (2a), and it is straightforward to check that Nordtvedt's expression is twice as large as this. However, this does not seem to affect any of Nordtvedt's results.

The Δ' term in equation (2b) can be made to vanish by the gauge transformation (Nordtvedt 1970):

$$x^{0'} = x^0-\frac{4\Delta'}{c}\frac{\partial}{\partial t}\sum_i m_i|\boldsymbol{r}-\boldsymbol{r}_i|,$$
$$x^{k'} = x^k, \qquad k = 1, 2, 3.$$

Apart from eliminating the Δ' term, this also has the effect of making the following changes in the other parameters:

$$\Delta \to \Delta - \Delta', \qquad \chi \to \chi - 8\Delta', \qquad \alpha'' \to \alpha'' - 2\Delta', \qquad \alpha''' \to \alpha''' - 8\Delta'.$$

This shows that the individual parameters do not have a direct physical significance, since they are coordinate dependent. Only certain combinations of them will be measurable. Choosing a particular value for Δ' is equivalent to adopting a gauge in which to work. Hence, from now on Δ' will be taken to be zero.

For each gravitational theory there is a set of values for the parameters in the post-newtonian metric. For example, in general relativity the parameters take the values:

$$\alpha = \beta = \gamma = \alpha' = \alpha'' = \alpha''' = \chi = \Delta = 1. \tag{3}$$

(see Einstein *et al* 1938, Eddington and Clark 1938).

On the other hand, in the Brans–Dicke theory the values are:

$$\begin{aligned} &\alpha = \beta = \alpha' = \alpha''' = \chi = 1, \\ &\gamma = \left(\frac{1+\omega}{2+\omega}\right), \\ &\Delta = \alpha'' = \left(\frac{3+2\omega}{4+2\omega}\right), \end{aligned} \tag{4}$$

where ω is the dimensionless constant of the theory, $\omega \simeq 5$, (see Estabrook 1969, Nordtvedt 1970), and in the Nordtvedt scalar–tensor theory the values are:

$$\begin{aligned} &\alpha = \alpha''' = \chi = 1, \\ &\gamma = \left(\frac{1+\omega}{2+\omega}\right), \\ &\Delta = \alpha'' = \left(\frac{3+2\omega}{4+2\omega}\right), \\ &\alpha' = 1 + \frac{2\omega'}{(4+2\omega)(3+2\omega)^2}, \\ &\beta = 1 + \frac{\omega'}{(4+2\omega)(3+2\omega)^2}, \end{aligned} \tag{5}$$

where $\omega' = \mathrm{d}\omega/\mathrm{d}\phi$ and ϕ denotes the scalar field (see Nordtvedt 1970).

It may be noted in passing that α must be equal to unity for any theory which is to agree with newtonian theory in the static weak field limit. But, for generality, α is retained explicitly.

3. The *n*-body equations

Only gravitational theories which can be expressed in geometrical terms are being considered here and so it will be assumed that the equations of motion of test particles are geodesics of the space–time. These are obtainable from the variation,

$$0 = \delta \int \mathrm{d}s,$$

which can be written

$$0 = \delta \int (g_{\mu\nu}\,\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu})^{1/2}$$

$$= \delta \int \mathrm{d}t(c^2 g_{00} + 2cg_{0k}v^k + g_{kl}v^k v^l)^{1/2}, \tag{6}$$

where t is the coordinate time and $v^k = \mathrm{d}x^k/\mathrm{d}t$. It is convenient to write equations (2) as

$$g_{00} = 1 + h_{00}^{(1)} + h_{00}^{(2)},$$

the superscript denoting powers of c^{-2},

$$g_{0k} = h_{0k},$$

and

$$g_{kl} = -(1-h)\delta_{kl}.$$

Then equation (6) becomes, to a sufficient approximation,

$$0 = \delta \int c\,\mathrm{d}t \left(1 - \frac{1}{2}\frac{v^2}{c^2} - \frac{1}{8}\frac{v^4}{c^4} + \frac{1}{2}(h_{00}^{(1)} + h_{00}^{(2)}) - \frac{1}{8}h_{00}^{(1)^2} + \frac{1}{c}h_{0k}v^k + \frac{v^2}{2c^2}(h + \tfrac{1}{2}h_{00}^{(1)})\right). \tag{7}$$

Carrying out the variation the following vector equation of motion is obtained:

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + \frac{1}{2c^2}\frac{\mathrm{d}}{\mathrm{d}t}(v^2\boldsymbol{v}) + \frac{\mathrm{d}}{\mathrm{d}t}(C(\boldsymbol{r})\boldsymbol{v}) - c\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{B}(\boldsymbol{r}) = -c^2\nabla A(\boldsymbol{r}) - c\nabla(\boldsymbol{B}\cdot\boldsymbol{v}) + \tfrac{1}{2}v^2\nabla C \tag{8}$$

where

$$A = \tfrac{1}{2}(h_{00}^{(1)} + h_{00}^{(2)}) - \tfrac{1}{8}h_{00}^{(1)^2},$$

$$\boldsymbol{B} = (h_{01}, h_{02}, h_{03}), \tag{9}$$

$$C = -(h + \tfrac{1}{2}h_{00}^{(1)}).$$

Using equations (2) (with $\Delta' = 0$), equations (9) become:

$$A = -\alpha\sum_i \frac{m_i}{|\boldsymbol{r}-\boldsymbol{r}_i|} + \left(\beta - \frac{\alpha^2}{2}\right)\left(\sum_i \frac{m_i}{|\boldsymbol{r}-\boldsymbol{r}_i|}\right)^2 + \alpha'\sum_i\sum_{j\neq i}\frac{m_i}{|\boldsymbol{r}-\boldsymbol{r}_i|}\frac{m_j}{|\boldsymbol{r}_i-\boldsymbol{r}_j|}$$
$$+\frac{\chi}{2c^2}\sum_i\frac{m_i(\boldsymbol{r}-\boldsymbol{r}_i)\cdot\boldsymbol{a}_i}{|\boldsymbol{r}-\boldsymbol{r}_i|} - \frac{2\alpha''}{c^2}\sum_i\frac{m_i v_i^2}{|\boldsymbol{r}-\boldsymbol{r}_i|} + \frac{\alpha'''}{2c^2}\sum_i\frac{\{(\boldsymbol{r}-\boldsymbol{r}_i)\cdot\boldsymbol{v}_i\}^2}{|\boldsymbol{r}-\boldsymbol{r}_i|^3},$$

$$\boldsymbol{B} = \frac{4\Delta}{c}\sum_i\frac{m_i\boldsymbol{v}_i}{|\boldsymbol{r}-\boldsymbol{r}_i|}$$

and

$$C = (\alpha + 2\gamma)\sum_i\frac{m_i}{|\boldsymbol{r}-\boldsymbol{r}_i|}.$$

Substituting these into equation (8) gives the desired equation of motion. By using the newtonian approximation, which in this case is

$$\boldsymbol{a}_{\mathrm{N}} = -\alpha \sum_i \frac{m_i c^2 (\boldsymbol{r}-\boldsymbol{r}_i)}{|\boldsymbol{r}-\boldsymbol{r}_i|^3},$$

where appropriate, the equation of motion can be written in the form

$$\begin{aligned}\boldsymbol{a} = &-\sum_i \frac{m_i(\boldsymbol{r}-\boldsymbol{r}_i)}{|\boldsymbol{r}-\boldsymbol{r}_i|^3}\Bigg(\alpha c^2-(2\beta+2\alpha\gamma)\sum_j \frac{m_j c^2}{|\boldsymbol{r}-\boldsymbol{r}_j|}-\alpha' \sum_{k\neq i}\frac{m_k c^2}{|\boldsymbol{r}_i-\boldsymbol{r}_k|}+2\alpha'' v_i^2+\gamma v^2-4\Delta \boldsymbol{v}\cdot\boldsymbol{v}_i \\ &-\frac{3}{2}\alpha'''\frac{\{(\boldsymbol{r}-\boldsymbol{r}_i)\cdot\boldsymbol{v}_i\}^2}{|\boldsymbol{r}-\boldsymbol{r}_i|^2}-\frac{\chi}{2}\{(\boldsymbol{r}-\boldsymbol{r}_i)\cdot\boldsymbol{a}_i\}\Bigg)+\sum_i\{(\alpha+2\gamma)\boldsymbol{v}-4\Delta\boldsymbol{v}_i\}\frac{m_i(\boldsymbol{r}-\boldsymbol{r}_i)\cdot(\boldsymbol{v}-\boldsymbol{v}_i)}{|\boldsymbol{r}-\boldsymbol{r}_i|^3} \\ &+\alpha\sum_i\frac{m_i\{(\boldsymbol{r}-\boldsymbol{r}_i)\cdot\boldsymbol{v}\}\boldsymbol{v}}{|\boldsymbol{r}-\boldsymbol{r}_i|^3}-\alpha'''\sum_i\frac{m_i\{(\boldsymbol{r}-\boldsymbol{r}_i)\cdot\boldsymbol{v}_i\}\boldsymbol{v}_i}{|\boldsymbol{r}-\boldsymbol{r}_i|^3}+\left(4\Delta-\frac{\chi}{2}\right)\sum_i\frac{m_i\boldsymbol{a}_i}{|\boldsymbol{r}-\boldsymbol{r}_i|}. \qquad (10)\end{aligned}$$

The newtonian value for $\boldsymbol{a}_i$ may be substituted in the right hand side of equation (10).

On substituting the values for the parameters given by equation (3), equation (10) gives the equation of motion in general relativity, in agreement with the result of Eddington and Clark (1938). The corresponding equation in the Brans–Dicke theory is obtained when the values of the parameters from equation (4) are substituted into equation (10). This agrees with the equation found by Brans (1962) and Estabrook (1969). Finally, the equation in the Nordtvedt theory can be obtained by using equation (5), namely,

$$\begin{aligned}\boldsymbol{a} = &-\sum_i \frac{m_i(\boldsymbol{r}-\boldsymbol{r}_i)}{|\boldsymbol{r}-\boldsymbol{r}_i|^3}\Bigg\{c^2-\left(\frac{6+4\omega}{2+\omega}+\frac{2\omega'}{(4+2\omega)(3+2\omega)^2}\right)\sum_j\frac{m_j c^2}{|\boldsymbol{r}-\boldsymbol{r}_j|} \\ &-\left(1+\frac{2\omega'}{(4+2\omega)(3+2\omega)^2}\right)\sum_{k\neq i}\frac{m_k c^2}{|\boldsymbol{r}_i-\boldsymbol{r}_k|}+\left(\frac{3+2\omega}{2+\omega}\right)v_i^2+\left(\frac{1+\omega}{2+\omega}\right)v^2 \\ &-\left(\frac{6+4\omega}{2+\omega}\right)\boldsymbol{v}\cdot\boldsymbol{v}_i-\frac{3}{2}\frac{\{(\boldsymbol{r}-\boldsymbol{r}_i)\cdot\boldsymbol{v}_i\}^2}{|\boldsymbol{r}-\boldsymbol{r}_i|^2}-\frac{1}{2}\{(\boldsymbol{r}-\boldsymbol{r}_i)\cdot\boldsymbol{a}_i\}\Bigg\}+\sum_i\frac{m_i(\boldsymbol{v}-\boldsymbol{v}_i)}{|\boldsymbol{r}-\boldsymbol{r}_i|^3} \\ &\times\left[(\boldsymbol{r}-\boldsymbol{r}_i)\cdot\left\{\left(\frac{6+4\omega}{2+\omega}\right)\boldsymbol{v}-\left(\frac{4+3\omega}{2+\omega}\right)\boldsymbol{v}_i\right\}\right]+\left(\frac{10+7\omega}{4+2\omega}\right)\sum_i\frac{m_i\boldsymbol{a}_i}{|\boldsymbol{r}-\boldsymbol{r}_i|}. \qquad (11)\end{aligned}$$

Note that equation (11) immediately reduces to the Brans–Dicke equation of motion when $\omega' = 0$, and to the general relativistic equation when $\omega' = 0$ and $\omega = \infty$.

Equation (10) gives the postnewtonian equation of motion of a test particle in the gravitational field of n point masses. For later applications it will be convenient to have the equation of motion of one of the masses in the gravitational field of the other $(n-1)$ masses. Equation (10) can be extended to cover this case by writing $\boldsymbol{r} = \boldsymbol{r}_i$, $\boldsymbol{v} = \boldsymbol{v}_i$ and $\boldsymbol{a} = \boldsymbol{a}_i$ and dropping any self-contributions (ie infinities) which occur on the right hand side of the equation. The equation then becomes

$$\begin{aligned}\boldsymbol{a}_i = &-\sum_{j\neq i}\frac{m_j(\boldsymbol{r}_i-\boldsymbol{r}_j)}{|\boldsymbol{r}_i-\boldsymbol{r}_j|^3}\Bigg(\alpha c^2-(2\beta+2\alpha\gamma)\sum_{k\neq i}\frac{m_k c^2}{|\boldsymbol{r}_i-\boldsymbol{r}_k|}-\alpha'\sum_{k\neq j}\frac{m_k c^2}{|\boldsymbol{r}_j-\boldsymbol{r}_k|}+2\alpha'' v_j^2+\gamma v_i^2-4\Delta\boldsymbol{v}_i\cdot\boldsymbol{v}_j \\ &-\frac{3}{2}\frac{\alpha'''\{(\boldsymbol{r}_i-\boldsymbol{r}_j)\cdot\boldsymbol{v}_j\}^2}{|\boldsymbol{r}_i-\boldsymbol{r}_j|^2}-\frac{\chi}{2}\{(\boldsymbol{r}_i-\boldsymbol{r}_j)\cdot\boldsymbol{a}_j\}\Bigg)+\sum_{j\neq i}\{(\alpha+2\gamma)\boldsymbol{v}_i-4\Delta\boldsymbol{v}_j\}\end{aligned}$$

$$\times \frac{m_j(\boldsymbol{r}_i-\boldsymbol{r}_j).(\boldsymbol{v}_i-\boldsymbol{v}_j)}{|\boldsymbol{r}_i-\boldsymbol{r}_j|^3}+\alpha\sum_{j\neq i}\frac{m_j\{(\boldsymbol{r}_i-\boldsymbol{r}_j).\boldsymbol{v}_i\}\boldsymbol{v}_i}{|\boldsymbol{r}_i-\boldsymbol{r}_j|^3}$$

$$-\alpha'''\sum_{j\neq i}\frac{m_j\{(\boldsymbol{r}_i-\boldsymbol{r}_j).\boldsymbol{v}_j\}\boldsymbol{v}_j}{|\boldsymbol{r}_i-\boldsymbol{r}_j|^3}+\left(4\Delta-\frac{\chi}{2}\right)\sum_{j\neq i}\frac{m_j\boldsymbol{a}_j}{|\boldsymbol{r}_i-\boldsymbol{r}_j|}. \tag{12}$$

4. The two-body problem

The formulae in general relativity for the two effects to be considered here are well known: (i) the advance of the periastron of the relative orbit of the two bodies is the same as that for a test particle orbiting a single body with a mass equal to the sum of the masses of the two bodies (Fock 1959); (ii) there is no secular acceleration of the centre of mass (Eddington and Clark 1938). Use is now made of a special case of equation (12) to calculate the corresponding formulae for the general metric of Nordtvedt.

Firstly, the equation of motion of each body in the gravitational field of the other has to be found. From equation (12) it follows that the acceleration of a body (subscript 1) in the field of another body (subscript 2) is

$$\boldsymbol{a}_1 = \frac{m_2\boldsymbol{r}_{12}}{r_{12}^3}\left(\alpha c^2-(2\beta+2\alpha\gamma)\frac{m_2c^2}{r_{12}}-\frac{\alpha' m_1c^2}{r_{12}}+2\alpha'' v_2^2+\gamma v_1^2-4\Delta\boldsymbol{v}_1.\boldsymbol{v}_2\right.$$
$$\left.-\frac{3}{2}\alpha'''\frac{(\boldsymbol{r}_{12}.\boldsymbol{v}_2)^2}{r_{12}^2}-\frac{\alpha\chi}{2}\frac{m_1c^2}{r_{12}}\right)+\{(\alpha+2\gamma)\boldsymbol{v}_1-4\Delta\boldsymbol{v}_2\}\frac{m_1v_{12}}{r_{12}^2}$$
$$-\alpha\frac{m_2(\boldsymbol{r}_{12}.\boldsymbol{v}_1)\boldsymbol{v}_1}{r_{12}^3}+\alpha'''\frac{m_2(\boldsymbol{r}_{12}.\boldsymbol{v}_2)\boldsymbol{v}_2}{r_{12}^3}-\alpha\left(4\Delta-\frac{\chi}{2}\right)\frac{m_1m_2c^2}{r_{12}^4}\boldsymbol{r}_{12}, \tag{13}$$

where $\boldsymbol{r}_{12} = (\boldsymbol{r}_2-\boldsymbol{r}_1)$, $r_{12} = |\boldsymbol{r}_2-\boldsymbol{r}_1|$, $\boldsymbol{v}_{12} = (\boldsymbol{v}_2-\boldsymbol{v}_1)$, $v_{12} = \mathrm{d}r_{12}/\mathrm{d}t$, and use has been made of the fact that $\boldsymbol{r}_{12}.\boldsymbol{v}_{12} = r_{12}v_{12}$. Also the newtonian approximation for $\boldsymbol{a}_2$ has been used, namely,

$$\boldsymbol{a}_2 = \alpha\frac{m_1c^2(\boldsymbol{r}_1-\boldsymbol{r}_2)}{|\boldsymbol{r}_1-\boldsymbol{r}_2|^3} = -\frac{\alpha m_1c^2}{r_{12}^3}\boldsymbol{r}_{12}.$$

The equation for $\boldsymbol{a}_2$ can be obtained by interchanging the subscripts 1 and 2.

The position of the centre of mass is defined by

$$\boldsymbol{r}_0 = \frac{1}{m}(m_1\boldsymbol{r}_1+m_2\boldsymbol{r}_2), \tag{14}$$

where $m = m_1+m_2$. Consequently,

$$\boldsymbol{a}_0 = \frac{1}{m}(m_1\boldsymbol{a}_1+m_2\boldsymbol{a}_2). \tag{15}$$

It is clear ırom the equations for $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ (equation (13) and its counterpart) that

$$\boldsymbol{a}_0 = \mathrm{O}(c^{-2}).$$

If the frame of reference is chosen to be instantaneously moving with the centre of mass, then the velocity of the centre of mass

$$\boldsymbol{v}_0 = \mathrm{O}(c^{-2}).$$

Hence, from equation (14),

$$\frac{1}{m}(m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2) = \mathrm{O}(c^{-2}).$$

Therefore,

$$\boldsymbol{v}_1 = -\frac{m_2}{m}\boldsymbol{v}_{12} + \mathrm{O}(c^{-2}),$$

and

$$\boldsymbol{v}_2 = \frac{m_1}{m}\boldsymbol{v}_{12} + \mathrm{O}(c^{-2}).$$

Using these, equation (13) can be further reduced to

$$\begin{aligned}\boldsymbol{a}_1 = \frac{m_2\boldsymbol{r}_{12}}{r_{12}^3}\bigg(&\alpha c^2 - (2\beta + 2\alpha\gamma)\frac{m_2 c^2}{r_{12}} - (\alpha' + 4\alpha\Delta)\frac{m_1 c^2}{r_{12}} \\ &-\frac{3}{2}\alpha'''\frac{m_1^2}{m^2}v_{12}^2 + \frac{1}{m^2}(2\alpha'' m_1^2 + 4\Delta m_1 m_2 + \gamma m_2^2)v_{12}^2\bigg) \\ &-\frac{m_2 v_{12}}{r_{12}^2}\left((\alpha + 2\gamma)\frac{m_2}{m} + 4\Delta\frac{m_1}{m} + \alpha\frac{m_2^2}{m^2} - \alpha'''\frac{m_1^2}{m^2}\right)\boldsymbol{v}_{12}.\end{aligned} \tag{16}$$

Again $\boldsymbol{a}_2$ is given by interchanging the subscripts 1 and 2.

5. Advance of the periastron

In order to calculate this effect the equation of the orbit of one of the bodies relative to the other has to be found. The relative acceleration of body 2 with respect to body 1 is, using equation (16) and its counterpart,

$$\begin{aligned}\boldsymbol{a}_{12} &= \boldsymbol{a}_2 - \boldsymbol{a}_1 \\ &= -\frac{\boldsymbol{r}_{12}}{r_{12}^3}\bigg(\alpha m c^2 - (2\beta + 2\alpha\gamma)\frac{c^2(m_1^2 + m_2^2)}{r_{12}} - (2\alpha' + 8\alpha\Delta)\frac{c^2 m_1 m_2}{r_{12}} \\ &\quad + \gamma v_{12}^2\frac{m_1^2 - m_1 m_2 + m_2^2}{m} + (2\alpha'' + 4\Delta)\frac{m_1 m_2}{m}v_{12}^2 - \frac{3}{2}\alpha'''\frac{m_1 m_2}{m}v_{12}^2\bigg) \\ &\quad + \frac{v_{12}}{r_{12}^2}\boldsymbol{v}_{12}\left((\alpha + 2\gamma)\frac{m_1^2 + m_2^2}{m} + 8\Delta\frac{m_1 m_2}{m} + \alpha\frac{m_1^2 - m_1 m_2 + m_2^2}{m} - \alpha'''\frac{m_1 m_2}{m}\right).\end{aligned} \tag{17}$$

It may be noted that, since $\boldsymbol{a}_{12} = 0$ wherever $\boldsymbol{r}_{12} = 0$ and $\boldsymbol{v}_{12} = 0$, the relative orbit is a plane to order c^{-2}. On changing to polar coordinates (r, θ) centred on body 1 and lying in this plane, the vector equation (17) can be written as the following two scalar equations:

$$\ddot{r}-r\dot{\theta}^2 = -\frac{1}{r^2}\Bigg(\alpha mc^2-(2\beta+2\alpha\gamma)c^2\frac{m_1^2+m_2^2}{r}-(2\alpha+8\alpha\Delta)c^2\frac{m_1m_2}{r}$$
$$-\frac{3}{2}\alpha'''\frac{m_1m_2}{m}\dot{r}^2+\gamma\frac{m_1^2+m_2^2}{m}(\dot{r}^2+r^2\dot{\theta}^2)+(2\alpha''+4\Delta-\gamma)\frac{m_1m_2}{m}(\dot{r}^2+r^2\dot{\theta}^2)\Bigg)$$
$$+\frac{\dot{r}^2}{r^2}\left((2\alpha+2\gamma)\frac{m_1^2+m_2^2}{m}+(8\Delta-\alpha'''-\alpha)\frac{m_1m_2}{m}\right), \tag{18}$$

and

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) = \frac{\dot{r}\dot{\theta}}{r}\left((2\alpha+2\gamma)\frac{m_1^2+m_2^2}{m}+(8\Delta-\alpha'''-\alpha)\frac{m_1m_2}{m}\right). \tag{19}$$

Integrating equation (19) in the newtonian approximation gives $r^2\dot{\theta} = h+\mathrm{O}(c^{-2})$, where h is a constant. Using this result, equation (19) can be integrated in the post-newtonian approximation as

$$r^2\dot{\theta} = h-\frac{h}{r}\left((2\alpha+2\gamma)\frac{m_1^2+m_2^2}{m}+(8\Delta-\alpha'''-\alpha)\frac{m_1m_2}{m}\right)+\mathrm{O}(c^{-4}). \tag{20}$$

On writing u for $1/r$, using equation (20) and the fact that

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}+u = \alpha\frac{mc^2}{h^2}+\mathrm{O}(c^{-2}), \tag{21}$$

equation (18) becomes,

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}+u = \alpha\frac{mc^2}{h^2}+u\left((4\alpha^2+2\alpha\gamma-2\beta)\frac{c^2}{h^2}(m_1^2+m_2^2)+\{2\alpha(4\Delta-\alpha'''-\alpha)-2\alpha'\}c^2\frac{m_1m_2}{h^2}\right)$$
$$+u^2\left(\gamma\frac{m_1^2+m_2^2}{m}+(2\alpha''+4\Delta-\gamma)\frac{m_1m_2}{m}\right)$$
$$+\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2\left(\gamma\frac{m_1^2+m_2^2}{m}+(2\alpha''+4\Delta-\gamma-\tfrac{1}{2}\alpha''')\frac{m_1m_2}{m}\right). \tag{22}$$

This is the equation of the relative orbit.

The motion of the periastron of this orbit will now be calculated. From equation (21) it follows that,

$$u = \alpha\frac{mc^2}{h^2}(1+\epsilon\cos\theta)+\mathrm{O}(c^{-2}), \tag{23}$$

where ϵ is the eccentricity of the orbit, and the origin of θ has been chosen so that the line $\theta = 0$ is directed along the line of the periastron of the relative orbit of the two bodies. On substituting equation (23) in the right hand side of equation (22), it is found that

$$\frac{d^2u}{d\theta^2}+u = \alpha\frac{mc^2}{h^2}+\text{constant terms of order } c^{-2}$$

$$+\epsilon\cos\theta\left(\alpha\frac{mc^4}{h^4}[(4\alpha^2+4\alpha\gamma-2\beta)(m_1^2+m_2^2) \ +\{2\alpha(8\Delta-\alpha+2\alpha''-\alpha'''-\gamma)-2\alpha'\}m_1m_2]\right)$$

$$+\text{terms in } \epsilon^2\sin^2\theta \text{ of order } c^{-2}. \tag{24}$$

By noting that the particular integral of

$$\frac{d^2u}{d\theta^2}+u = A\cos\theta$$

is

$$u = \tfrac{1}{2}A\theta\sin\theta,$$

it can be deduced from equation (24) that,

$$u = \alpha\frac{mc^2}{h^2}\left\{1+\epsilon\cos\theta+\epsilon\left(\{2\alpha(\alpha+\gamma)-\beta\}\frac{c^2}{h^2}(m_1^2+m_2^2)\right.\right.$$

$$\left.\left.+\{\alpha(8\Delta-\alpha+2\alpha''-\alpha'''-\gamma)-\alpha'\}\frac{c^2}{h^2}m_1m_2\right)\theta\sin\theta\right\}$$

$$+\text{constant and periodic terms of order } c^{-2}+O(c^{-4}).$$

From which it follows that the angular advance of the periastron, per revolution, is

$$2\pi\frac{m^2c^2}{h^2}\left(\{2\alpha(\alpha+\gamma)-\beta\}\frac{m_1^2+m_2^2}{m^2}+\{\alpha(8\Delta-\alpha+2\alpha''-\alpha'''-\gamma)-\alpha'\}\frac{m_1m_2}{m^2}\right). \tag{25}$$

It can easily be seen that, if the relation

$$\alpha(8\Delta-\alpha+2\alpha''-\alpha'''-\gamma)-\alpha' = 2\{2\alpha(\alpha+\gamma)-\beta\} \tag{26}$$

holds between the parameters, then the expression in the large bracket of equation (25) is a perfect square. If this is so, then the motion of the periastron is of the same type as in general relativity, that is, the advance is the same as for a particle orbiting a single body of mass m_1+m_2. Equations (4) and (5) show that relation (26) holds in both the Brans–Dicke and Nordtvedt scalar–tensor theories. It therefore follows that in all three theories the advance of the periastron can be written in the form $2\pi c^2K(m_1+m_2)^2/h^2$, where $K = 3$ in general relativity, $K = (4+3\omega)/(2+\omega)$ in the Brans–Dicke theory and

$$K = \frac{4+3\omega}{2+\omega}-\frac{\omega'}{(4+2\omega)(3+2\omega)^2}$$

in the Nordtvedt theory.

6. Motion of the centre of mass

The secular acceleration of the centre of mass in the two-body problem will now be calculated to order c^{-2}. From equation (15) the acceleration of the centre of mass $\boldsymbol{a}_0$ is

$$\boldsymbol{a}_0 = \frac{1}{m}(m_1\boldsymbol{a}_1 + m_2\boldsymbol{a}_2).$$

On substituting from equation (16), and its counterpart for $\boldsymbol{a}_2$, it is found that

$$\boldsymbol{a}_0 = \frac{m_1m_2(m_1-m_2)}{m^2}\left\{\frac{\boldsymbol{r}_{12}}{r_{12}^3}\left((2\beta+2\alpha\gamma-\alpha'-4\alpha\Delta)\frac{mc^2}{r_{12}}+(2\alpha''-\gamma)\boldsymbol{v}_{12}^2-\frac{3}{2}\alpha'''v_{12}^2\right)\right.$$
$$\left.+(2\alpha+2\gamma-4\Delta+\alpha''')\frac{v_{12}\boldsymbol{v}_{12}}{r_{12}^2}\right\}+O(c^{-4}). \qquad (27)$$

By changing to the polar coordinates centred on body 1, defining $x = r\cos\theta$ and $y = r\sin\theta$ with similar definitions for x_0 and y_0, using equations (20) and (23) and the fact that

$$\dot{x} = -\alpha\frac{mc^2}{h}\sin\theta + O(c^{-2}),$$

and

$$\dot{y} = \alpha\frac{mc^2}{h}(\epsilon+\cos\theta)+O(c^{-2}),$$

equation (27) can be written as

$$\frac{d\dot{x}_0}{d\theta} = m_1m_2(m_1-m_2)\alpha\frac{c^4}{h^3}\{(2\beta+2\alpha\gamma-\alpha'-4\alpha\Delta)\cos\theta(1+\epsilon\cos\theta)$$
$$+\alpha(2\alpha''-\gamma)\cos\theta(1+\epsilon^2+2\epsilon\cos\theta)-\tfrac{3}{2}\alpha\alpha'''\epsilon^2\sin^2\theta\cos\theta$$
$$-\alpha(2\alpha+2\gamma-4\Delta+\alpha''')\epsilon\sin^2\theta\}+O(c^{-4}), \qquad (28)$$

and

$$\frac{d\dot{y}_0}{d\theta} = m_1m_2(m_1-m_2)\alpha\frac{c^4}{h^3}\{(2\beta+2\alpha\gamma-\alpha'-4\alpha\Delta)\sin\theta(1+\epsilon\cos\theta)$$
$$+\alpha(2\alpha''-\gamma)\sin\theta(1+\epsilon^2+2\epsilon\cos\theta)-\tfrac{3}{2}\alpha\alpha'''\epsilon^2\sin^3\theta$$
$$+\alpha(2\alpha+2\gamma-4\Delta+\alpha''')(\epsilon+\cos\theta)\sin\theta\}+O(c^{-4}). \qquad (29)$$

The secular increase in the velocity of the centre of mass per revolution is found by integrating equations (28) and (29) from 0 to 2π. It is easily seen that there is no change in the y_0 component. The increase in the x_0 component is

$$m_1m_2(m_1-m_2)\frac{c^4}{h^3}\epsilon\pi\{\alpha(2\beta-2\alpha\gamma-\alpha'+4\alpha\alpha''-\alpha\alpha'''-2\alpha^2)\}. \qquad (30)$$

By the choice of coordinate system the secular acceleration (equation (30)) is directed towards the periastron of the smaller or the larger body as the expression in the curly brackets is positive or negative.

As Eddington and Clark (1938) have pointed out a zero result for this secular acceleration is desirable on general physical grounds. Equation (3) shows that the acceleration is zero in general relativity in agreement with Eddington and Clark's result. Equations (4) and (5) show that a zero result is also obtained in the Brans–Dicke and Nordtvedt theories. However, this zero value is certainly not a conclusion to be expected in every gravitational theory since equation (30) involves 6 of the 8 postnewtonian parameters. If a result of this type is to be regarded as desirable (ie as a necessary condition to be satisfied by a gravitational theory) then it places a definite restriction on the types of postnewtonian metric which are acceptable.

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